# ON THE STABILITY OF STEADY HELICAL MOTIONS IN A FLUID OF A BODY BOUNDED BY A MULTIPLY CONNECTED SURFACE

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The stability of steady motions of a rigid body has been the subject of repeated study. Grammel [1] and Rumiantsev [2] studied the stability of uniform rotations of a heavy rigid body with a fixed point. Kuz'min [3] and Irtegov [4] have accounted for the Newtonian force field. Mozalevskaia [5] investigated the stability of uniform rotations of a gyrostat in a potential field. Stability of helical motions in an unbounded ideal incompressible fluid was dealt with by Liapunov [6] under the assumption that the surface bounding the body was singly connected. Kharlamov generalized the equations of motion of a rigid body in a fluid to the case of a body bounded by a multiply connected surface [7] and in addition generalized the analogy noted by Steklov in [8]. Using this analogy we can obtain not only the equations of motion of a heavy rigid body with a fixed point as a particular case of the Kharlamov equations, but also the more general equations of motion of a gyrostat in a Newtonian field. It follows that the conditions of stability for the above cases should also follow from the conditions of stability of motion of a body in a fluid, postulated in [7]. This generalization lends therefore some interest to the analysis of the stability of helical motions in a fluid of a body bounded by a multiply connected surface.

**1.** Differential equations of motion are [7]

$$dP_1/dt = (P_2 + \lambda_2)\omega_3 - (P_3 + \lambda_3)\omega_2 + (u_3 - \mu_3)R_2 - (u_2 - \mu_2)R_3 \qquad (1.1)$$

$$dR_1 / dt = \omega_3 R_2 - \omega_2 R_3 \qquad (123)$$

where the remaining four equations are obtained by the cyclic interchange of indices denoted by the symbol (123). Moreover,

$$\omega_i = \frac{\partial T}{\partial P_i}, \qquad u_j = \frac{\partial T}{\partial R_j} \qquad (i, j = 1, 2, 3) \qquad (1.2)$$

The following three integrals are well known

$$T - \mu_i R_i = h \qquad (1.3)$$

$$(P_1 + \lambda_1) R_1 + (P_2 + \lambda_2) R_2 + (P_3 + \lambda_3) R_3 = m, \qquad R_1^2 + R_2^2 + R_3^2 = R^2 \qquad (1.4)$$

Let us now suppose that the origin of a moving coordinate system associated with the body is situated at its center and, that its axes coincide with the principal axes of the impulsive momentum ellipsoid. Then the kinetic energy of the system will be

$$2T = \mathrm{S}\,[a_{11}P_1^2 + 2c_{11}P_1R_1 + 2c_{12}(P_1R_2 + P_2R_1) + 2b_{12}R_1R_2 + b_{11}R_1^2] \qquad (1.5)$$

where $S$ denotes summation of three terms obtained from the expression under the $S$-sign by cyclic interchange of the indices 1, 2 and 3.

**2.** By the Routh criterion [9] we need the minimum value of the integral (1.3) with (1.4) satisfied, in order to obtain the sufficient conditions of stability of motion of a body in a fluid. We shall investigate the stability of motion relative to $P_1, P_2, P_3, R_1, R_2, R_3$. Let us construct the Lagrangian

$$F = h - sm - {}^1/_2\sigma R^2 \tag{2.1}$$

where $s$ and $\sigma$ are the Lagrange multipliers and let us write the necessary conditions for the extremum of the function $F$ with respect to $P_1, \ldots, R_3$

$$\partial F / \partial P_i = 0, \qquad \partial F / \partial R_j = 0$$

This, together with (1.2), yields

$$\omega_i = sR_i, \qquad u_i - \mu_i = s(P_i + \lambda_i) + \sigma R_i \tag{2.2}$$

Substituting now $\omega_i$ and $u_i - \mu_i$ into (1.1) we see, that the function $F$ assumed its extremal value at the steady helical motions. First three Eqs. of (2.2) give the variables $P_i$ in the terms of $R_j$

$$P_1 = -1/a_{11} [(c_{11} - s)R_1 + c_{12}R_2 + c_{13}R_3] \qquad (123) \tag{2.3}$$

Here and in the following the symbol (123) means that the remaining two equations are obtained by the cyclic interchange of indices.

Let $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ denote the incremental growth of $P_1, \ldots, R_3$ respectively, in the perturbed motion. Relations (2.1), (1.3), (1.4) and (1.5) yield

$$2\delta^2 F = \mathrm{S}[a_{11}\xi^2_1 + (b_{11} - \sigma)\eta^2_1 + 2c_{12}(\xi_1\eta_2 + \xi_2\eta_1) + 2b_{12}\eta_1\eta_2 + 2(c_{11} - s)\xi_1\eta_1] + \ldots$$

$$(P_1 + \lambda_1)\eta_1 + (P_2 + \lambda_2)\eta_2 + (P_3 + \lambda_3)\eta_3 + R_1\xi_1 + R_2\xi_2 + R_3\xi_3 + \ldots = \delta m$$

$$R_1\eta_1 + R_2\eta_2 + R_3\eta_3 + \ldots = {}^1/_2\delta R^2$$

Performing the substitution

$$\zeta_1 = \xi_1 + \frac{c_{11} - s}{a_{11}}\eta_1 + \frac{c_{12}}{a_{11}}\eta_2 + \frac{c_{13}}{a_{11}}\eta_3 \qquad (123)$$

with (2.3) taken into account, we have

$$A_{11} = b_{11} - \frac{(c_{11} - s)^2}{a_{11}} - \frac{c_{12}^2}{a_{22}} - \frac{c_{13}^2}{a_{33}} \qquad (123)$$

$$\frac{1}{2}A_{12} = b_{12} - \frac{(c_{11} - \sigma)\, c_{12}}{a_{11}} - \frac{(c_{21}\, c_{22} - s)}{a_{22}} - \frac{c_{31}c_{32}}{a_{33}} \qquad (123)$$

$$X_1 = 2\frac{c_{11} - s}{a_{11}} R_1 + \left(\frac{c_{12}}{a_{11}} + \frac{c_{12}}{a_{22}}\right) R_2 + \left(\frac{c_{13}}{a_{11}} + \frac{c_{13}}{a_{33}}\right) R_3 - \lambda_1 \qquad (123)$$

and then we obtain

$$2\delta^2 F = \mathrm{S}\,[(A_{11} - \sigma)\eta_1^2 + A_{12}\eta_1\eta_2 + a_{11}\zeta_1^2] - \tag{2.4}$$

$$-X_1\eta_1 - X_2\eta_2 - X_3\eta_3 + R_1\zeta_1 + R_2\zeta_2 + R_3\zeta_3 = \delta m$$

$$R_1\eta_1 + R_2\eta_2 + R_3\eta_3 = {}^1/_2\delta R^2 \tag{2.5}$$

Sufficient conditions of stability of helical motions are obtained from the conditions of the positive definiteness of the quadratic form (2.4) with the conditions (2.5). The form $\delta^2 F$ will, under the conditions (2.5), be positive definite if the principal diagonal minors of the order higher than three of the following determinant [10] are positive

$$\begin{vmatrix}
0 & 0 & -X_1 & -X_2 & -X_3 & R_1 & R_2 & R_3 & \delta m \\
0 & 0 & R_1 & R_2 & R_3 & 0 & 0 & 0 & {}^1/_2\delta R^2 \\
-X_1 & R_1 & (A_{11} - \sigma) & A_{12} & A_{13} & 0 & 0 & 0 & 0 \\
-X_2 & R_2 & A_{21} & (A_{22} - \sigma) & A_{23} & 0 & 0 & 0 & 0 \\
-X_3 & R_3 & A_{31} & A_{32} & (A_{33} - \sigma) & 0 & 0 & 0 & 0 \\
R_1 & 0 & 0 & 0 & 0 & a_{11} & 0 & 0 & 0 \\
R_2 & 0 & 0 & 0 & 0 & 0 & a_{22} & 0 & 0 \\
R_3 & 0 & 0 & 0 & 0 & 0 & 0 & a_{33} & 0 \\
\delta m & {}^1/_2\delta R^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix}$$

The minors are given by

$$\Delta_5 = M > 0, \qquad \Delta_8 = a_{11}a_{22}a_{33}\,[M + \Theta N) > 0$$

$$\Delta_6 = a_{11}M + R_1^2 N > 0, \quad \Delta_9 = a_{11}a_{22}a_{33}\,[N\,(\delta m)^2 + L\delta m\delta R^2 + {}^1/_4 K(\delta R^2)^2] > 0$$

$$\Delta_7 = a_{11}a_{22}M + (a_{11}R_2^2 + a_{22}R_1^2)N > 0$$

$$\Theta = R_1^2 \,/\, a_{11} + R_2^2 \,/\, a_{22} + R_3^2 \,/\, a_{33}$$

$$M = \mathrm{S}\ [(A_{11} - \sigma)(R_2X_3 - R_3X_2)^2 + 2A_{12}\,(R_2X_3 - R_3X_2)(R_3X_1 - R_1X_3)]$$

$$N = \mathrm{S}\ \{[(A_{22} - \sigma)(A_{33} - \sigma) - A_{23}^2]R_1^2 + 2[A_{31}A_{23} - A_{12}\,(A_{33} - \sigma)]R_1R_2\}$$

$$L = \mathrm{S}\ \{[(A_{22} - \sigma)(A_{33} - \sigma) - A_{23}^2]R_1X_1 + 2[A_{31}A_{23} - A_{12}\,(A_{33} - \sigma)](R_1X_2 + R_2X_1)\}$$

$$K = \mathrm{S}\,\{[(A_{22} - \sigma)(A_{33} - \sigma) - A_{23}^2]X^2_1 + 2[A_{31}A_{23} - A_{12}\,(A_{33} - \sigma)]X_1X_2\} + \\ + \{(A_{11} - \sigma)[(A_{22} - \sigma)(A_{33} - \sigma) - A_{23}^2] + A_{12}[A_{31}A_{23} - A_{12}(A_{33} - \sigma)] + \\ + A_{31}\,[A_{23}A_{12} - A_{31}\,(A_{22} - \sigma)]\}\,\theta$$

Conditions of stability depend on the restrictions imposed on the perturbations, and may vary from case to case.

1) Perturbations leave the integrals (1.4) unaltered; sufficient condition of stability is, that

$$M > 0, \qquad M + \Theta N > 0$$

2) If $\delta m \neq 0$ and $\delta R^2 = 0$, then the conditions become

$$M > 0, \qquad N > 0$$

3) If $\delta m = 0$ and $\delta R^2 \neq 0$, then

$$M > 0, \qquad M + \Theta N > 0, \qquad K > 0$$

is sufficient.

4) Let us now consider the case when the perturbations are unrestricted (both integrals $m$ and $R^2$ vary).

Putting $\varkappa = \delta m / \delta R^2$ we shall write the condition $\Delta_0 > 0$ as

$$N\varkappa^2 + L\varkappa + {}^1/_4 K > 0 \tag{2.6}$$

When $N > 0$, the condition (2.6) holds for any $\varkappa$, provided that

$$L^2 - KN < 0$$

If $N < 0$, (2.6) will hold if $\varkappa$ lies within the interval ( $\varkappa_1$, $\varkappa_2$ ) where $\varkappa_1$ and $\varkappa_2$ are the roots of the following Eq.

$$N\varkappa^2 + L\varkappa + {}^1/_4 K = 0$$

Finally, if we assume that the integrals (1.4) vary freely, then we must only consider the case $N > 0$;

$$M > 0, \qquad N > 0, \qquad L^2 - KN < 0$$

will then represent the sufficient conditions of stability.

**3.** The following analogy was noted by Kharlamov in [7]. Putting in (1.5)

$$a_{ij} = \begin{cases} A_i^{-1} & (i = j) \\ 0 & (i \neq j) \end{cases}, \qquad b_{ij} = \begin{cases} \varepsilon A_i & (i = j) \\ 0 & (i \neq j) \end{cases}, \qquad c_{ij} = 0, \qquad P_i = A_i\omega_i$$

we obtain, from (1.1),

$$A_1 d\omega_1 \,/\, dt = (A_2 - A_3)\,(\omega_2\omega_3 - \varepsilon R_2R_3) + \lambda_2\omega_3 - \lambda_3\omega_2 + \mu_2R_3 - \mu_3R_2$$

$$dR_1 \,/\, dt = \omega_3R_2 - \omega_2R_3 \qquad (123)$$

which formally coincide with the equations of motion of a gyrostat in a central Newtonian field. These equations possess the following known integrals

$$T - \mu_iR_i = H, \quad (A_1\omega_1 + \lambda_1)R_1 + (A_2\omega_2 + \lambda_2)R_2 + (A_3\omega_3 + \lambda_3)R_3 = m$$

$$R_1^2 + R_2^2 + R_3^2 = R^2 = 1$$

We easily see that we can obtain the conditions of stability of uniform rotation of a gyrostat, from the conditions of stability of the steady helical motions of the body in a fluid. The following two conditions are sufficient to ensure the stability of steady rotations of a gyrostat under arbitrary perturbations:

$$M > 0, \qquad N > 0$$

while the perturbations leaving the value of the surface integral unchanged will only require

$$M > 0, \qquad M + (A_1R_1^2 + A_2R_2^2 + A_3R_3^2)N > 0 \tag{3.1}$$

$$M = \Omega S (\rho - A_1)[2\omega (A_2 - A_3)R_2R_3 - R_2\lambda_3 + R_3\lambda_2]^2, \quad N = \Omega^2 S (\rho - A_2)(\rho - A_3)R_1^2 \tag{3.2}$$

as sufficient conditions. Here $\omega = s$ is the angular velocity of a body rotating about the vertical, while $\Omega$ and $\rho$ are given by

$$\Omega = \omega^2 - \varepsilon, \qquad \sigma = -\Omega\rho$$

For a rigid body ($\lambda_i = 0$) the stability conditions obtained from (3.1) coincide with the conditions of Kuz'min [3]. When the perturbations are arbitrary, then we find that the sufficient stability conditions are more strict for a rigid body, and are given by

$$M > 0, \qquad N > 0 \qquad (M = 4\Omega\omega^2 S (\rho - A_1)(A_2 - A_3)^2R_2^2R_3^2)$$

where $N$ has the form of (3.2). These conditions coincide with those obtained from the conditions given [5].

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